

# CRITERIA FOR CONFORMAL INVARIANCE OF $(0, 2)$ MODELS

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It is argued that many linear  $(0,2)$  models flow in the infrared to conformally invariant solutions of string theory. The strategy in the argument is to show that the effective space-time superpotential must vanish because there is no place where it can have a pole. This conclusion comes from either of two different analyses, in which the Kahler class or the complex structure of the gauge bundle is varied, while keeping everything else fixed. In the former case, we recover from the linear sigma model the usual simple pole in the  $\overline{27}^3$  Yukawa coupling but show that an analogous pole does not arise in the couplings of gauge singlet modes.

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## 1. Introduction

$N = 1$  spacetime supersymmetry in string theories requires at least  $(0,2)$  supersymmetry on the worldsheet [1]. The bulk of effort in studying string vacua has, however, gone into  $(2,2)$  theories, in part because they are easier to study; in particular, sufficient conditions for conformal invariance are better understood. For instance, Dixon [2] showed using the superconformal Ward identities that  $(2,2)$ -preserving gauge singlet fields are moduli of string theory, a result which holds to all orders in string loop perturbation theory. The proof used the left-moving supersymmetries crucially, and indeed it was noted early on [3][4] that unlike  $(2,2)$  models,  $(0,2)$  sigma models are susceptible to worldsheet instanton corrections to the vacuum energy. Certain  $(0,2)$  sigma models have been argued to avoid destabilization by instantons [5][6][7][8], but the conditions required for these arguments have been very special.

Linear sigma models give a new way to study the parameter spaces of  $(0,2)$  and  $(2,2)$  theories [9]. A very large class of models can be represented by linear sigma models; this includes, for instance, arbitrary complete intersections in weighted projective spaces (or more general toric varieties). Distler and Kachru have used linear sigma models to study interesting classes of  $(0,2)$  vacua in their Landau-Ginzburg phases [10] and showed, for example, that a wide class of  $E_6$ -invariant  $(0,2)$  Landau-Ginzburg models were conformally invariant [11].

In this paper we give new methods to argue for conformal invariance of those  $(0,2)$  models that can be represented by linear sigma models. We focus for illustration on the  $E_6$ -invariant  $(0,2)$  deformations of the quintic, but we believe that the considerations are far more general.

The basic idea is to study the space-time superpotential as a function of the moduli on which it depends; insofar as the moduli space is compact and the superpotential is not identically zero, the superpotential must have poles somewhere.<sup>1</sup> Poles can only arise when the parameters are taken to values at which the compactness of the target space is lost because some boson fields can go to infinity. At large field strength, quantum corrections to the classical theory are small and calculable, so the possible poles can be located. Moreover, the polar parts of the various couplings can be explicitly computed. We will analyze the

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<sup>1</sup> A holomorphic function without poles on a compact complex manifold would have to be constant. The superpotential is not really a holomorphic function but a section of a line bundle of negative curvature [12], so it cannot even be constant: if there are no poles, it must vanish.

behavior as either the Kahler class or the complex structure of the bundle is varied and show in each case that (i) the linear sigma model gives a natural compact parameter space, (ii) the places where the sigma model breaks down can be concretely described, and (iii) the relevant couplings do not have poles at those places.

The paper is organized as follows. In section 2, we explain heuristically why linear sigma models may be special and why, in fact, conformal invariance should possibly be understood on more elementary grounds than we will explore here. In section 3, we analyze the singularity locations of the linear sigma model. In section 4, we perform computations near the singularity of the Kahler moduli space, recovering the usual pole in Yukawa couplings of charged fields and showing that no pole arises for the  $E_6$  singlet modes. Together with the compactness of the Kahler moduli space of the linear sigma model, this gives our first argument for conformal invariance. In section 5, we analyze the behavior as a function of the complex structure of the manifold and bundle, showing again the usual pole for the charged fields but the absence of the pole for the singlets. The argument here can be carried out as in section 4. In section 6 we briefly discuss more general models.

## 2. More Elementary Considerations

In trying to compute non-zero space-time superpotentials for moduli fields in  $(0, 2)$  linear sigma models, we encountered certain difficulties which make us suspect that the problem of conformal invariance in these models should perhaps be understood in a more elementary way than we will eventually present. We will here sketch some of these issues.

To compute the spacetime superpotential from a worldsheet theory, one needs to know what interactions are determined by a given superpotential. The general structure of Yukawa couplings coming from a superpotential  $W$  is as follows (using the conventions of [13]):

$$L = \dots - e^{-\exp(K/2)} \left\{ W \bar{\psi}_a \bar{\sigma}^{ab} \bar{\psi}_b + \frac{i}{2} \sqrt{2} D_i W \chi^i \sigma^a \bar{\psi}_a + \frac{1}{2} D_i D_j W \chi^i \chi^j + h.c. \right\} \quad (2.1)$$

Here the  $\psi_a$  is the gravitino, the  $\chi^i$  are fermions from chiral multiplets (their scalar partners will be called  $A^i$ ), and  $K$  is the Kahler potential. One way to study the superpotential is through its covariant derivatives using for example the last term in (2.1); this is what we will do in the bulk of this paper. But a simpler option seems to present itself here: we could compute  $W$  directly by evaluating the  $\langle \bar{\psi}_a \bar{\sigma}^{ab} \bar{\psi}_b A^i \rangle$  coupling, which would yield

$-e(\frac{1}{2}\frac{\partial K}{\partial A^i})\exp(K/2)W$  evaluated at the vacuum expectation values of the scalar fields  $A_i$ . To compute this amplitude in string theory we compute the correlation function of the corresponding vertex operators in the two-dimensional quantum field theory.

This correlation function can be restricted by considering the right-moving  $U(1)$   $R$  charge. As will be discussed more fully in section 3, the transformation properties of a mode under spacetime gauge and Lorentz symmetries determines its  $U(1)$  charges. The gravitino generates spectral flow, and (in the canonical ghost picture) has internal right  $U(1)$  charge  $q_R^{gravitino} = 3/2$ . Spacetime scalar fields  $A^i$  have  $q_R^{scalar} = \pm 1$  (aside from the dilaton, which comes from the gravitational sector of the theory and has zero internal right  $U(1)$  charge). Therefore the above amplitude does not conserve the internal right-moving  $U(1)$  and the worldsheet correlation function vanishes trivially.

One might question whether the above argument is circular. We have used some of the usual properties of string solutions and vertex operators, and perhaps the argument only proves that when one does have a string solution, then the space-time superpotential is  $W = 0$ . In general, if one does not have a classical solution of string theory, one does not know in what kind of world-sheet theory the computation is to be performed.

This is where linear sigma models may be relevant. Linear sigma models give definite quantum field theories that flow from known (free) fixed points in the ultraviolet, and which (under mild and familiar conditions) at least appear to flow to conformal field theories in the infrared. Moreover, many of their essential properties are known. For instance, the argument above mainly used the  $R$  symmetry, which is valid even away from criticality in appropriate linear models.

It might appear that if one does not have a conformal field theory, one does not know what is meant by the gravitino and scalar vertex operators in the above computation. However, those vertex operators are all chiral primaries (of the right-movers), which makes it possible to identify them with states in the  $\overline{Q}_+$  cohomology (vertex operators of the half-twisted model) even away from criticality. Thus it appears that the above computation of  $W = 0$  makes sense without an *a priori* assumption of conformal invariance.

By contrast, consider a general Calabi-Yau manifold  $X$  with a stable holomorphic vector bundle  $V$ . One can try to use this data to determine a conformal field theory, and this is at least approximately valid, modulo instanton corrections, near the field theory limit. However, one does not know any *exact* quantum field theory, conformally invariant or not, determined by this data. In particular, one does not know if these theories can be “cut off” in a way that preserves the right-moving  $R$  symmetry. If in fact, because

of instanton effects, a general  $(X, V)$  corresponds to a theory that is *not* conformally invariant, and flows to *weak* coupling in the infrared, then we may be in difficulty: it might be that any ultraviolet theory that would flow to this renormalization group orbit is strongly coupled, and we have no way to know if the  $R$  symmetry can be maintained.

Thus, we believe that a possible picture is that a general  $(X, V)$  does not correspond to any cut-off independent,  $R$ -invariant quantum field theory in the ultraviolet or any conformal field theory in the infrared. On the other hand, those particular  $(X, V)$  that can be realized as linear sigma models – though vast in number, they are a tiny fraction of abstract  $(X, V)$ 's<sup>2</sup> – do come from definite, known  $R$ -invariant theories in which the above computation, giving  $W = 0$ , can be performed.

Since we are in fact not sure whether the above line of reasoning is valid, we will proceed in the rest of this paper to a more technical discussion. In passing, we will note also that we could study  $(0, 2)$  linear sigma models from the point of view of analyzing non-perturbatively the non-renormalization of the world-sheet superpotential and twisted superpotential (by methods that are familiar in four dimensions [15]). Though we believe that such an analysis would go through, we will not go down that road since the implications of such an analysis for conformal invariance and the space-time superpotential are not clear to us. If such a relation could be understood, this route might again give a more direct and elementary treatment of conformal invariance of linear  $(0, 2)$  models than we will give in this paper.

### 3. Singularities

We will illustrate our reasoning with a familiar example of a Calabi-Yau manifold – a quintic hypersurface in  $\mathbf{CP}^4$ . Since we want to be able to consider certain  $(0, 2)$  deformations of the usual  $(2, 2)$  model, we formulate it as a  $(0, 2)$  linear sigma model. This can be done as follows (see section 6.2 of [9]). We work in  $(0, 2)$  superspace with fermionic coordinates  $\theta^+$  and  $\bar{\theta}^+$  and bosonic coordinates  $y^\alpha$ ,  $\alpha = 1, 2$ . The goal is to incorporate the parameters determining the size and complex structure of the manifold  $X$

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<sup>2</sup> The same assertion is true for  $(2, 2)$  models: though a huge number of algebraic varieties come from linear sigma models, generic ones do not. Calabi-Yau manifolds that do not come from linear sigma models are perhaps less familiar as they require more sophisticated algebraic geometry for their construction and analysis. See [14], pp. 103, 204 for a construction of one such example.

and those determining the complex structure of the stable holomorphic vector bundle  $V$  as coupling constants of a linear sigma model which at large radius reduces to the familiar nonlinear sigma model on  $(X, V)$  in the infrared. In particular, the imaginary part of the Kahler parameter,  $r$ , will arise as the coefficient of a Fayet-Iliopolous  $D$ -term of a world-sheet  $U(1)$  gauge group. In terms of the gauge-covariant superspace derivative  $\mathcal{D}_+$  defined in ([9], equations 6.2 and 6.4) the field strength superfield for the  $U(1)$  gauge group is  $\Upsilon = [\overline{\mathcal{D}}_+, \mathcal{D}_0 - \mathcal{D}_1]$ . In Wess-Zumino gauge it becomes  $\Upsilon = i\overline{\mathcal{D}}_+ V + \partial_- \overline{\mathcal{D}}_+ \overline{\Psi}$  where the superfields  $\Psi = \theta^+ \overline{\theta}^+ (v_0 + v_1)$  and  $V = v_0 - v_1 - 2i\theta^+ \overline{\lambda}_- - 2i\overline{\theta}^+ \lambda_- + 2\theta^+ \overline{\theta}^+ D$  organize the gauge field  $v_\mu$  and the gaugino  $\lambda_-$  into a  $(0, 2)$  gauge supermultiplet. There is also a gauge neutral chiral multiplet  $\Sigma' = \sigma + \sqrt{2}\theta^+ \lambda_+ - i\theta^+ \overline{\theta}^+ (D_0 + D_1)\sigma$  (which in a  $(2, 2)$  language would combine with  $\Upsilon$  into a twisted chiral superfield).

The complex structure and bundle parameters will come from superpotential terms. These will involve six chiral superfields

$$\Phi^I = \phi^I + \sqrt{2}\theta^+ \psi_+^I - i\theta^+ \overline{\theta}^+ (D_0 + D_1)\phi^I, \quad I = 0, \dots, 5 \quad (3.1)$$

To construct the gauge-invariant superpotential of interest we will take  $\Phi^0 = P$  to have charge  $-5$  and the  $\Phi^i = S^i$ ,  $i = 1 \dots 5$  to have charge 1. The bosonic components of these superfields are called  $p$  and  $s^i$  respectively. In addition, there are six fermionic multiplets

$$\Psi_-^I = \psi_-^I - \sqrt{2}\theta^+ G^I - i\theta^+ \overline{\theta}^+ (D_0 + D_1)\psi_-^I - 2i\overline{\theta}^+ Q^I \Sigma' \Phi^I \quad (3.2)$$

with the same gauge charges  $Q^I$  as the corresponding  $\Phi^I$ . Here  $G^I$  is an auxiliary field which gets integrated out in favor of the  $(0, 2)$  superpotential term  $\overline{J}_I$  which is introduced below. (In a  $(2, 2)$  language, the  $\Phi^I$  and  $\Psi_-^I$  would combine into ordinary chiral superfields.) The  $\Psi_-$  obey a chirality condition

$$\overline{\mathcal{D}}_+ \Psi_-^I = E^I \quad (3.3)$$

where  $E^I$  are holomorphic functions of the chiral superfields  $\Phi^J$ . For  $(0, 2)$  models that arise as deformations of  $(2, 2)$  models,

$$E^I = 2iQ^I \Sigma' \Phi^I. \quad (3.4)$$

To actually write a solution of heterotic string theory, one would also include additional degrees of freedom, such as free fermions, to represent a left-moving  $SO(10) \times E_8$  current algebra; we will for the time being not need to make this explicit.

The Lagrangian consists of the standard flat kinetic terms together with the following. There is a  $U(1)$   $D$ -term with coefficient  $r$  and a  $\theta$ -term; together these can be written in the gauge-invariant form

$$\begin{aligned} L_{D,\theta} &= \frac{t}{4} \int d^2y \, d\theta^+ \, \Upsilon \Big|_{\bar{\theta}^+=0} + h.c. \\ &= \int d^2y \left( -rD + \frac{\theta}{2\pi} v_{01} \right) \end{aligned} \quad (3.5)$$

where  $t = ir + \frac{\theta}{2\pi}$ . There is also a  $(0, 2)$  superpotential

$$L_J = -\frac{1}{\sqrt{2}} \int d^2y \, d\theta^+ \, \Psi_-^I J_I \Big|_{\bar{\theta}^+=0} + h.c. \quad (3.6)$$

Here

$$J_0(\Phi^I) = G(S^i) \quad (3.7)$$

where  $G = G_{ijklm} S^i S^j S^k S^l S^m$  is the defining polynomial for the quintic hypersurface, and

$$J_i = F_{i,j_1 j_2 j_3 j_4} S^{j_1} S^{j_2} S^{j_3} S^{j_4} P, \quad i = 1, \dots, 5 \quad (3.8)$$

is required to obey

$$S^i J_i = 5P J_0 = 5PG(S^i). \quad (3.9)$$

This ensures (using (3.4) and the values of the charges) that  $\sum_I E^I J_I = 0$ , a necessary condition for  $(0, 2)$  supersymmetry. The model actually has  $(2, 2)$  supersymmetry if and only if

$$J_i = P \frac{\partial G}{\partial S^i}. \quad (3.10)$$

It is convenient to set

$$J_i \equiv P \tilde{J}_i(S^j) \quad (3.11)$$

Departing from (3.10) breaks  $(2, 2)$  supersymmetry to  $(0, 2)$  and has the effect of perturbing the tangent bundle of the quintic to a more general bundle  $V$ . (Since the perturbed bundle has rank three, the space-time gauge group is still  $E_6 \times E_8$ .) This is seen as follows. Massless left-moving fermions satisfy  $J^i \psi_{-,i} = 0$  for  $i = 1, \dots, 5$ . A vector in  $V$  can be described by five  $v^i$ , subject to the equivalence  $v^i \simeq v^i + \lambda s^i$ , and satisfying  $J_i v^i = 0$ . These conditions are compatible on the hypersurface  $G = 0$  by virtue of (3.9). If we decompose  $\tilde{J}_i$  as  $\tilde{J}_i = \frac{\partial G}{\partial s^i} + G_i$  for some quartics  $G_i$  satisfying  $G_i s^i = 0$ , then the 224 parameters in  $G_i$  are the moduli that break  $(2, 2)$  down to  $(0, 2)$ .

The model has a left-moving  $U(1)$  symmetry, which we will call  $U(1)_L$  (it is actually part of a left-moving  $E_6$  current algebra), and a right-moving  $U(1)$   $R$  symmetry, which we will call  $U(1)_R$ . The charges carried by the various fields are as shown in Table 1.

**Table 1: U(1) Charges**

<u>Fields</u>	<u><math>(J_L, J_R)</math></u>	<u>Fields</u>	<u><math>(J_L, J_R)</math></u>
$\psi_+^i$	$(1/5, -4/5)$	$\lambda_-$	$(0, 1)$
$\psi_-^i$	$(-4/5, 1/5)$	$\lambda_+$	$(1, 0)$
$\psi_+^0$	$(0, -1)$	$\sigma$	$(-1, 1)$
$\psi_-^0$	$(-1, 0)$	$s^i$	$(1/5, 1/5)$
$p$	$(0, 0)$		

The parameter space consists of  $t$  and the  $F_{i,j_1j_2j_3j_4}$ . If we subtract the 25 linear redefinitions of the  $s^i$ , these contain 326 independent parameters. At large radius these can be distinguished as one Kahler parameter, 101 complex structure deformations, and 224 deformations of the holomorphic structure of  $V$ . On the (2,2) locus  $t$  and the 101  $G_{ijklm}$  are true moduli and the left-moving supersymmetry provides an invariant distinction between them:  $Q_-$  annihilates the mode conjugate to  $t$  while  $\overline{Q}_-$  annihilates the modes conjugate to the  $G$ 's. For the general (0,2) situation, there is no such algebraic distinction between the various modes, and it is not clear whether theories parametrized by these variables actually correspond in the infrared to conformal field theories; exploring this point is precisely the goal of our investigation.

The parameter space we have just found is compact, or at least has a natural compactification. Consider first the  $t$  variable. Because of the invariance under  $\theta \rightarrow \theta + 2\pi$ , that is,  $t \rightarrow t + 1$ , the natural variable is  $q = e^{2\pi i t}$ . It appears from this definition that  $q$  runs over the whole complex plane except for the origin. However, it is natural to add two points: the point  $q = 0$  corresponds to the infinite radius or field theory limit of the theory, and  $q = \infty$  is the Landau-Ginzburg point. (These assertions can be seen semiclassically [9].) With these points included, the Kahler moduli space is a copy of  $\mathbf{CP}^1$ , which is compact.

The other moduli can be treated as follows. Rescaling the  $F_{i,jklm}$  and the  $G_{ijklm}$  by a common complex number  $\lambda$  can be absorbed in a rescaling of the  $\Phi_i$  and the  $\Lambda_i$  by  $\lambda^{-\frac{1}{5}}$  up to a change in the  $D$  terms of the theory. It is believed that the change in the  $D$  terms does not affect the infrared fixed point of the theory. Accepting this and dividing by the overall scaling, the parameter space of these modes is a complex projective space  $\mathbf{CP}^{349}$ ;



this is compact. (We could also absorb 24 of these 349 parameters in a linear redefinition of the  $s^i$  of determinant 1, so there are only 325 physical deformations, as discussed above. Dividing just by scaling is more convenient and sufficient to prove compactness.) Of course, at some points in this  $\mathbf{CP}^{349}$ , the model will become singular; this is part of what will be analyzed later.

The bosonic potential energy of the theory (computed as in section 6.2 of [9]) is

$$\begin{aligned} U(\phi_I) &= \frac{e^2}{2} \left( \sum_I Q_I |\phi_I|^2 - r \right)^2 + \sum_I Q_I^2 |\phi_I|^2 |\sigma|^2 + \sum_I |J_I|^2 \\ &= \frac{e^2}{2} \left( \sum_i |s_i|^2 - 5|p|^2 - r \right)^2 + |\sigma|^2 \left( \sum_i |s_i|^2 + 25|p|^2 \right) + |G|^2 + \sum_i |J_i|^2 \end{aligned} \quad (3.12)$$

The first term on the right hand side comes from integrating out the auxiliary field  $D$ . The theory can be studied semiclassically at large  $|r|$ ; for  $r \gg 0$  we find the Calabi-Yau phase and for  $r \ll 0$  the Landau-Ginzburg phase at low energy.

More generally, we are interested in the infrared behavior of the theory for general  $t$  and  $F_{i,j_1 j_2 j_3 j_4}$ . Since we do not have direct access to the infrared limit, we mostly restrict attention to computations which are invariant under renormalization group flow. This sector of the theory is conveniently packaged in the various (quasi-)topological twisted theories [16][17][18][19]. In these theories, the stress tensor is shifted by

$$T_{\alpha\beta} \rightarrow T'_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{4}(\epsilon^\gamma_\alpha \partial_\gamma J_\beta + \epsilon^\gamma_\beta \partial_\gamma J_\alpha) \quad (3.13)$$

where  $J$  is  $J_R + J_L$  for the  $A$ -model,  $J_R - J_L$  for the  $B$ -model, and  $J_R$  for the half-twisted model<sup>3</sup>. This has the effect of shifting the spins of all fields by half their  $J$  charge. In particular,  $\overline{Q}_+$  becomes a scalar charge and plays the role of a BRST operator, so that physical states of the twisted theory are elements of  $\overline{Q}_+$  cohomology.

In the  $(0,2)$  case, the various twisted models are not topological field theories, but they are still conformally invariant, as we will now explain. Because the action and measure are invariant under the symmetry generated by  $\overline{Q}_+$ , there is an interesting sector of the theory in which one considers only operators annihilated by  $\overline{Q}_+$ . Such a correlation functions vanishes if one of the operators is  $\overline{Q}_+$ -trivial,  $\alpha = \{\overline{Q}_+, \beta\}$  (and one does not meet anomalies from surface terms). We also have the relation  $\text{Tr}(T) = T_{+-} = \{\overline{Q}_+, \dots\}$ ,

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<sup>3</sup> Actually we are free to shift  $J$  by the gauge current  $\mathcal{J}$ , which is  $\{\overline{Q}_+, \dots\}$ ; we will find it convenient to do so in section 4.

a statement that holds in all three models because of the underlying supersymmetry. These properties ensure formally that all these models are conformally invariant, since insertions of  $\text{Tr}(T)$  vanish in correlation functions of physical observables. Actually, these theories have a much stronger property:  $T_{++}$  is similarly  $\{\overline{Q}_+, \dots\}$ , so that correlation functions of  $\overline{Q}_+$ -invariant operators vary holomorphically in addition to being conformally invariant.

We are interested in correlation functions of vertex operators in the physical model. The relation between the physical and twisted models has to do with spectral flow [20][21][22]. Shifting the spins as described above is realized in the path integral by adding appropriate couplings of the fields to the spin connection. This will be implemented explicitly for the computations of interest for us in section 3. The space-time degrees of freedom undergo spectral flow at the same time, accomplished by insertions of space-time spin operators  $S_\alpha$  or  $S_{\dot{\beta}}$ .

As explained in [23], the  $J_L$  and  $J_R$  values classify the transformation properties of the states under the space-time gauge and Lorentz groups. The theory can be decomposed into the space-time (“external”) and internal c=9 parts, so that  $h^{tot} = h^{int} + h^{ext}$ ,  $q^{tot} = h^{int} + q^{ext}$  etc. From the right-moving N=2 algebra we have  $h_R^{tot} \geq \frac{1}{2}|q_R|$ , and for right-moving NS states  $h_R^{tot} = \frac{1}{2} \Rightarrow q_R = \pm 1$ . A scalar in space-time is invariant under the external U(1) so that  $q_R^{int} = q_R = \pm 1$ . Its fermion partner has  $q_R = \pm \frac{1}{2}$ .

The left U(1) is part of the spacetime gauge group. We will be interested in three types of three-point functions involving the generations, antigerations, and  $E_6$  singlets:  $\overline{\mathbf{27}}^3$ ,  $\mathbf{27}^3$ , and  $\mathbf{S}^3$ .<sup>4</sup> As just explained, twisting the model is equivalent to inserting spectral flow generators. In particular, the twisting by  $J_R$  turns the two fermion vertex operators into boson vertex operators. For the  $A$  and  $B$  models we also twist by  $\pm J_L$ , which takes us from a spinor representation of  $SO(10)$  with half-integral  $J_L$  (so that the *total* left-moving U(1) charge is integral) to a scalar or vector representation of  $SO(10)$  with integral  $J_L$ . In particular, under  $SO(10) \times U(1)$ , the  $\overline{\mathbf{27}}$  of  $E_6$  decomposes as  $\overline{\mathbf{27}} = \overline{\mathbf{16}}_{1/2} \oplus \mathbf{10}_{-1} \oplus \mathbf{1}_2$  and left spectral flow takes us from  $\mathbf{10}_{-1}$  to  $\overline{\mathbf{16}}_{1/2}$  to  $\mathbf{1}_2$ .

More explicitly the  $\overline{\mathbf{27}}^3$  amplitude can be performed in the  $A$ -model as follows:

$$< V_F^{\overline{\mathbf{16}}_{1/2}} V_B^{\mathbf{10}_{-1}} V_F^{\overline{\mathbf{16}}_{1/2}} >_{phys} \propto < V_B^{-1} V_B^{-1} V_B^{-1} >_{A, internal} \quad (3.14)$$

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<sup>4</sup> Note that in our notation,  $\mathbf{S}$  refers to *any*  $E_6$  singlet, not only those which are associated to  $H^1(\text{End } V)$  at large radius.

where the proportionality factor is obtained by evaluating the  $SO(10)$  correlator in free field theory, and where the superscripts on the internal vertex operators indicate left  $U(1)$  charges. Similarly, for the  $\mathbf{27^3}$  amplitude we have the  $B$ -model computation

$$< V_F^{\mathbf{16}-1/2} V_B^{\mathbf{10}_1} V_F^{\mathbf{16}-1/2} >_{phys} \propto < V_B^1 V_B^1 V_B^1 >_{B, internal} \quad (3.15)$$

Finally, for the  $\mathbf{S^3}$  amplitude we obtain a half-twisted model computation:

$$< V_F^0 V_B^0 V_F^0 >_{phys} = < V_B^0 V_B^0 V_B^0 >_{H, internal} \quad (3.16)$$

We will introduce in section 4 the exact forms of the linear sigma model vertex operators we will use in the various twisted computations.

### 3.1. Location Of Singularities

In order to pin down the space-time superpotential, we would like to understand where this two-dimensional theory can become singular. The basic strategy in analyzing singularities is as follows. Suppose (this is a simplification) that for generic couplings the bosonic potential (3.12) had the property of going to infinity as the fields go to infinity – in any direction in field space. This growth of the potential at infinity would ensure the convergence of the path integral. Singularities could then only arise for parameters such that (in some direction in field space) the growth of the potential at infinity is lost.

Since we want to locate the singularities in the *quantum* theory, what is really relevant here is the large field behavior of the quantum effective potential. Happily, because of the super-renormalizability of the linear sigma model, the semiclassical approximation is valid for large fields and (except for a slight shift due to a one loop effect that will be noted in section 4) the classical potential can be used in locating singularities.

A more serious problem in implementing this program is that the assumption we made about the potential was too optimistic. Looking at (3.12), we see that by setting  $\Phi^I = 0$ , we can take  $\sigma$  to infinity for only a finite cost in action; the action in this limit is  $e^2 A r^2 / 2$ , with  $A$  the area of the surface. (Actually, as we will discuss more thoroughly in section 4, there is an additional term  $e^2 (A/2) (\theta/2\pi)^2$  coming from the fact that in two dimensions a non-zero  $\theta$  induces a stable background electric field [24].) This complicates the discussion, as we will analyze in section 4, but only in a relatively mild way, roughly because in the infrared limit (relevant to the conformal field theory we really wish to study)  $A$  is going to infinity.

However, it is clear that at some value of  $t$  for  $r$  near 0, the barrier against going to large  $\sigma$  might altogether vanish. We will determine in section 4 precisely where this occurs and show that a singularity actually does occur at the value of  $t$  in question.

Singularities might also occur as functions of the  $J_I$ . This will occur if the  $J_I$  are such that the equations  $J_I = 0$  can be obeyed with  $p \neq 0$  and some of the  $s^i$  non-zero. Then, taking  $p$  and the appropriate  $s^i$  to infinity (in such a ratio as to ensure vanishing of the  $D$  term), one has vanishing classical potential, so one should expect a divergence of the path integral. We will study this region in section 5, showing that it is isomorphic to the situation near  $t = 0$  and does give a singularity.

Once we locate the singularities, we can determine which physical quantities diverge there. In particular, by showing that the  $\mathbf{S}^3$  couplings have no singularities, we will be able to deduce – since the parameter spaces are compact – that they vanish identically.

#### 4. Behavior near $r = \theta = 0$

We have just noted that the bosonic potential has a dangerous behavior for  $\phi^I = 0, \sigma \rightarrow \infty$ , and a still more dangerous behavior when in addition  $t$  is near 0. We will in this section analyze the behavior of the path integral in this region. The analysis is tractable because most of the fields (including the  $\phi^I$ ) have masses proportional to  $|\sigma|$  for  $\sigma \rightarrow \infty$ , and the light degrees of freedom become free in this region.

As discussed in the last section, the linear sigma model becomes singular at  $t = ir + \theta/2\pi = 0$  due to the vanishing of the potential of  $\sigma$ , which produces an unbounded zero mode and a continuum of gapless excitations. By studying the behavior of gauge-singlet correlation functions near  $t = 0$ , we can determine the order of the pole in the spacetime superpotential. Before doing this we will study the well-known  $\overline{\mathbf{27}}^3$  coupling in this region in order to check our methods and provide a quick derivation of the simple pole in this amplitude. Then we will investigate whether the  $\mathbf{S}^3$  correlators can diverge at this locus.

To begin with we are interested in the Yukawa coupling  $\langle V_F^{\overline{\mathbf{27}}} V_B^{\overline{\mathbf{27}}} V_F^{\overline{\mathbf{27}}} \rangle$ . Specifically we will study here the component  $\langle V_F^{\overline{\mathbf{16}}_{1/2}} V_B^{\mathbf{10}_{-1}} V_F^{\overline{\mathbf{16}}_{1/2}} \rangle$ . This way the fermions are in the (R, R) sector: the right-moving spinor ground state ensures that the vertex operator describes a spacetime fermion, and the left-movers lie in the Ramond sector so that the operator transforms in the spinor ( $\mathbf{16}$ ) representation of  $\text{SO}(10)$ . The boson vertex operator

for the  $\mathbf{10}_{-1}$  component of the  $\overline{\mathbf{27}}$  is  $\chi_j \sigma$  for  $j = 1, \dots, 10$  where the  $\chi_j$  are free left-moving fermions in the vector representation of  $\text{SO}(10)$ . In the (2,2) model  $\sigma$  is the lowest component of the twisted chiral superfield  $\Sigma = \sigma - i\sqrt{2}\theta^+ \bar{\lambda}_+ - i\sqrt{2}\theta^- \lambda_- + \sqrt{2}\theta^+ \bar{\theta}^- (D - iv_{01}) + \dots$ , so on the (2,2) locus  $\sigma$  is related as it should be by a left-moving supersymmetry transformation to the Kahler modulus  $t$ .

It is straightforward to check that  $V_B^{\mathbf{10}_{-1}} = \sigma$  is normalized in the standard fashion at large radius, so that in particular  $\langle \sigma^3 \rangle$  has no pole at large radius [25]. (This will also help determine the normalization of singlet vertex operators later.) For  $r \gg 0$  we have  $\sum_i |s^i|^2 = r$  and  $p = 0$ . The part of the Lagrangian coupling  $\sigma$  to the  $\psi_\pm^i$  and  $\bar{\psi}_\pm^i$  is

$$\mathcal{L}_\sigma = \int d^2y \left( r|\sigma|^2 + \sqrt{2} \sum_i \bar{\psi}_+^i \bar{\sigma} \psi_-^i + \sqrt{2} \sum_i \bar{\psi}_-^i \sigma \psi_+^i \right). \quad (4.1)$$

In the  $A$ -model the fermions which have zero modes are the three linear combinations of  $\bar{\psi}_+^i$  and of  $\psi_-^i$  which are superpartners of the three combinations of  $s^i$  which are tangent to the manifold. Since  $r$  is very large,  $\sigma$  is very massive, and we can integrate it out using its equations of motion. The  $\bar{\sigma}$  equation gives

$$\sigma = -\frac{\sqrt{2}}{r} \bar{\psi}_+^i \psi_-^i. \quad (4.2)$$

Consider the transformation laws of  $s^i$  and  $\bar{s}^i$  under  $\bar{Q}_+$  and  $Q_-$ :

$$\begin{aligned} \{\bar{Q}_+, s^i\} &= 0, & \{\bar{Q}_+, \bar{s}^i\} &= -\sqrt{2} \bar{\psi}_+^i; \\ \{Q_-, s^i\} &= \sqrt{2} \psi_-^i, & \{Q_-, \bar{s}^i\} &= 0. \end{aligned} \quad (4.3)$$

These suggest that we interpret  $\bar{Q}_+$  as the  $\bar{\partial}$  operator and  $Q_-$  as the  $\partial$  operator of the space of  $s^i$ . Then  $\bar{\psi}_+^i$  is identified with  $d\bar{s}^i$  and  $\psi_-^i$  is identified with  $ds^i$ . Then from (4.2) we see that at large radius  $\sigma$  becomes identified with a (1,1) form, normalized so (as the volume of  $X$  is of order  $r^3$  and  $\sigma$  is of order  $1/r$ ) that  $\int_X \sigma \wedge \sigma \wedge \sigma$  is independent of  $r$ .

To evaluate this correlator as it stands we would need an expression for the covariant fermion vertex operators in the linear sigma model. It is easier to instead work in a “twisted” model as described in the previous section. In particular,

$$\langle V_F^{\overline{\mathbf{16}}_{1/2}} V_B^{\mathbf{10}_{-1}} V_F^{\overline{\mathbf{16}}_{1/2}} \rangle_{phys} \propto \langle \sigma \sigma \sigma \rangle_A \quad (4.4)$$

where the subscript  $A$  refers to the analogue of the fully twisted “ $A$ -model”, in which the spins of all fields are shifted by  $\frac{1}{2}J = -\frac{1}{2}(J_L + J_R) + \frac{1}{5}Q$ ; here  $Q$  is the gauge charge which we can add for convenience as indicated in the footnote following equation (3.13). This allows us to compute using only the simple internal bosonic vertex operator  $\sigma$ . The spin zero fields are now  $\lambda_+$ ,  $\bar{\lambda}_-$ ,  $\psi_{+,0}$ ,  $\bar{\psi}_-^0$ ,  $\psi_-^i$ ,  $\bar{\psi}_+^i$ ,  $s^i$ , and  $\sigma$ . ( $p$  is then twisted and will therefore later have no zero mode.)

#### 4.1. An Anomaly

Since the path integral has a dangerous behavior for  $\sigma \rightarrow \infty$  with  $\phi^I$  near zero, we first want to analyze the behavior in this regime to make sure that the path integral does converge. We will see that it does converge but poorly enough that there are anomalies in some formal assertions made in section 3. Once these anomalies are understood, we will be in a position to proceed to more precise computations.

We will first compute the contribution of the large  $\sigma$  region to (4.4). We can integrate out the  $\Phi_I$  multiplets, which have masses of order  $\sigma$ . This will give an effective action for the light modes,  $v_\mu$ ,  $\lambda_-$ ,  $\bar{\lambda}_-$ ,  $\sigma$ ,  $\lambda_+$ , and  $\bar{\lambda}_+$ , which we will then study more carefully. In fact, for large  $\sigma$ ,  $(2, 2)$  supersymmetry is restored and the light modes can be conveniently organized in a  $(2, 2)$  superfield

$$\begin{aligned} \Sigma = & \sigma - i\sqrt{2}\theta^+\bar{\lambda}_+ - i\sqrt{2}\bar{\theta}^-\lambda_- + \sqrt{2}\theta^+\bar{\theta}^-(D - iv_{01}) - i\bar{\theta}^-\theta^-(\partial_0 - \partial_1)\sigma \\ & - i\bar{\theta}^+\theta^+(\partial_0 + \partial_1)\sigma + \sqrt{2}\bar{\theta}^-\theta^+\theta^-(\partial_0 - \partial_1)\bar{\lambda}_+ + \sqrt{2}\theta^+\theta^-\bar{\theta}_+(\partial_0 + \partial_1)\bar{\lambda}_- \\ & - \theta^+\bar{\theta}^-\theta^-\bar{\theta}^+(\partial_0^2 - \partial_1^2)\sigma \end{aligned} \quad (4.5)$$

In fact, because of the large masses and the super-renormalizable nature of the theory, the only relevant term is the one loop contribution that comes by integrating out the  $\phi^i$ . This gives (upon setting  $v_\mu \equiv 0$  in the last step, anticipating a stationary point with this property):

$$\begin{aligned} & \int \prod_i d^2s^i d^2p \exp - \int Ds^i D\bar{s}^i + Dp D\bar{p} + D(|s^i|^2 - 5|p|^2) + |\sigma|^2(|s^i|^2 + 25|p|^2) \\ & = \left( \prod_i \det \frac{1}{-(\partial_\mu + iv_\mu)^2 + D + |\sigma|^2} \right) \det \frac{1}{-(\partial_\mu - 5iv_\mu)^2 - 5D + 25|\sigma|^2} \\ & = \exp \left\{ \int \frac{d^2k}{(2\pi)^2} \left[ -5 \ln(k^2 + D + |\sigma|^2) - \ln(k^2 - 5D + 25|\sigma|^2) \right] \right\} \end{aligned} \quad (4.6)$$

Performing the integral with constant  $D$  and  $\sigma$  and exponentiating this back into the action, the effective equation of motion for the auxiliary field  $D$  (including also the classical contribution  $-Dr + D^2/2e^2$ ) is

$$D - r + \frac{5}{2\pi} \ln 5 + \frac{5}{4\pi} \ln\left(\frac{|\sigma|^2 - \frac{D}{5}}{|\sigma|^2 + D}\right) = 0 \quad (4.7)$$

Expanding in  $1/|\sigma|^2$  we find

$$D(1 - \frac{3}{2\pi|\sigma|^2}) = r - \frac{5}{2\pi} \ln 5 \quad (4.8)$$

We can reproduce this equation of motion for  $D$  in the following (2,2) superspace Lagrangian (which was derived in the context of the  $\mathbf{CP}^n$  model in [26]):

$$S_{eff} = \int d^2y \left( \int d^4\theta \left[ -\left(\frac{1}{4e^2}\right) \bar{\Sigma} \Sigma - \frac{3}{8\pi} \ln \Sigma \ln \bar{\Sigma} \right] + \frac{it_R}{2\sqrt{2}} \int d\theta^+ d\bar{\theta}^- \Sigma - \frac{i\bar{t}_R}{2\sqrt{2}} \int d\theta^- d\bar{\theta}^+ \bar{\Sigma} \right) \quad (4.9)$$

which is good for large  $\sigma$ . Here the (2,2) twisted chiral superfield  $\Sigma$  is given by (4.5), and  $t_R$  is a renormalized Kahler parameter,

$$t_R = t - \frac{5}{2\pi} \ln 5. \quad (4.10)$$

There are also  $D$ -independent terms in (4.6). They largely cancel, because of supersymmetry, against similar  $D$ -independent terms in the fermion determinants. (If one can set  $D$  to zero, then the region of large  $\sigma$  and zero  $\phi^I$  has unbroken supersymmetry.) The cancellation is complete except for the “constant” modes of the various fields, the zero modes of the kinetic energy. The complex bosons  $s^i$  each have a constant mode (*not*  $p$ , which is twisted by virtue of its gauge charge); such a complex bosonic “zero mode” contributes a factor of  $1/\bar{\sigma}\sigma$  to the path integral. For the fermions, as we are in genus zero, only the components  $\psi_+^0$ ,  $\bar{\psi}_-^0$ ,  $\bar{\psi}_+^i$ , and  $\psi_-^i$  that are twisted to have spin zero have such “zero modes”; each pair  $(\bar{\psi}_+^i, \psi_-^i)$  contributes a factor of  $\bar{\sigma}$  and the pair  $(\psi_+^0, \bar{\psi}_-^0)$  contributes a factor of  $\sigma$ . Multiplying these factors, the net  $D$  independent contribution to the path integral is a factor of  $1/\sigma^4$ . This can be interpreted as the contribution of an effective action

$$\delta S_{eff} = \int d^2y \sqrt{g} \frac{\hat{R}}{4\pi} [-4 \ln \sigma] \quad (4.11)$$

This form of the effective action could also be predicted by studying the underlying anomalies.

To evaluate

$$\int d^2\sigma d\bar{\lambda}_- d\lambda_+ \sigma^3 e^{-S_{eff}} \quad (4.12)$$

we proceed as follows. To absorb the fermion zero-modes here we bring down a factor of  $\int d^2y \frac{2\sqrt{2}}{\sigma\bar{\sigma}^2} (D - iv_{01}) \bar{\lambda}_- \lambda_+$  from the component expansion of the  $\ln \Sigma \ln \bar{\Sigma}$  term in the action (4.9). We now must recall a subtlety alluded to earlier involving the  $\theta$ -angle in two-dimensional electrodynamics: as a function of  $\theta$ , the expectation value of  $v_{01}$  is [24]  $-e^2\theta/2\pi$  (with  $\theta$  understood to be in the range  $|\theta| \leq \pi$ ). Then (4.12) becomes

$$(-iAt_R) \int d^2\sigma \frac{2\sqrt{2}}{\sigma^2\bar{\sigma}^2} e^{-\frac{Ae^2|t_R|^2}{2}} \quad (4.13)$$

with  $A$  the area of the world-sheet. So the integral converges for  $\sigma \rightarrow \infty$  despite the poor behavior of the classical potential in that region. However, the convergence is relatively slow, and it is a familiar story that when physical amplitudes are given by integrals that barely converge, there often are anomalies in formal arguments about their properties. In this case, a possible anomaly suggests itself: despite the formal argument in section 3 for conformal invariance, the above formula certainly suggests that  $\langle \sigma^3 \rangle$  may be  $A$  dependent.

Some more information is needed to check this, since (4.13) is only valid for large  $\sigma$ . However, we will now demonstrate that there is indeed an anomaly in the formal argument for conformal invariance. We recall that the formal argument depends on writing the trace of the stress-tensor as

$$Tr(T) = \{\bar{Q}_+, B\}, \quad (4.14)$$

where  $B$  is a component of the supercurrent. Given (4.14), one tries to prove conformal invariance as follows. Under a conformal transformation  $\delta g = \alpha g$ , the change in  $\langle \sigma^3 \rangle$  is proportional to

$$< Tr(T(x)) \sigma \sigma \sigma > = < \{\bar{Q}_+, B(x)\} \sigma \sigma \sigma > = 0 \quad (4.15)$$

The last step really involves integration by parts on the  $\sigma$  plane.

To make this systematic, we interpret the zero modes via differential forms on the  $\sigma$  plane. Consider the transformation laws under the supersymmetry transformation generated by  $\bar{Q}_+$ :  $\delta\sigma = 0$  and  $\delta\bar{\sigma} = -i\sqrt{2}\bar{\epsilon}_- \lambda_+$ . We see that in the large  $\sigma$  region, it is natural



to identify  $\overline{Q}_+$  with the  $\overline{\partial}$  operator of the  $\sigma$  plane and  $\lambda_+$  with  $d\overline{\sigma}$ . In particular, acting on the zero-modes of the worldsheet fields,

$$\overline{Q}_+ = \sqrt{2}\lambda_+ \frac{\partial}{\partial \overline{\sigma}} + \dots \quad (4.16)$$

where  $\dots$  refers to terms which disappear when  $t \rightarrow 0$ . Suppose we want to compute  $\langle \{\overline{Q}_+, \Lambda\} \rangle$  for some  $\Lambda$ . In the  $A$ -model  $\lambda_+$  and  $\overline{\lambda}_-$  have spin zero. After integrating out the nonzero modes we are left with  $\langle \{\overline{Q}_+, \Lambda\} \rangle$  as a function of the fermion zero modes  $\lambda_+$  and  $\overline{\lambda}_-$  as well as the bosonic zero modes  $\sigma$  and  $\overline{\sigma}$ . Integrating out the fermion zero-modes picks out the component  $\eta$  of  $\Lambda$  which multiplies the fermion zero-mode  $\overline{\lambda}_-$ . Then we are left with an integral

$$\langle \{\overline{Q}_+, \eta\} \rangle = \int_U d^2\sigma \frac{\partial \eta}{\partial \overline{\sigma}} = \int_{\partial U} d\sigma \eta, \quad (4.17)$$

where  $U$  is the  $\sigma$  plane and  $\partial U$  is a circle at infinity. We can schematically write  $\int_{\partial U} \eta = \langle \Lambda \rangle'$  where  $\langle \dots \rangle'$  is a modified correlation function in which one suppresses the zero mode of the fermion field  $\overline{\lambda}_-$  and integrates only over a circle at infinity in the  $\sigma$  plane. With this understood, (4.15) becomes

$$\langle T\sigma^3 \rangle = \langle b\sigma^3 \rangle'. \quad (4.18)$$

where  $b$  is the component of  $B$  which multiplies  $\overline{\lambda}_-$ . Thus, we have a precise framework for detecting a possible anomaly.

From the effective action, one finds that

$$\begin{aligned} Tr(T) = & \frac{-3\sqrt{2}\overline{\lambda}_-\lambda_+}{4\pi\sigma\overline{\sigma}^2}(D - iv_{01}) + \frac{3\sqrt{2}\overline{\lambda}_+\lambda_-}{4\pi\overline{\sigma}\sigma^2}(D + iv_{01}) \\ & + \frac{3\overline{\lambda}_+\lambda_-\overline{\lambda}_-\lambda_+}{2\pi|\sigma|^4} - rD + \frac{D^2}{2}\left(1 - \frac{3}{2\pi|\sigma|^2}\right) \end{aligned} \quad (4.19)$$

Hence using the supersymmetry transformation laws  $\{\overline{Q}_+, \overline{\sigma}\} = i\lambda_+\sqrt{2}$ ,  $\{\overline{Q}_+, \overline{\lambda}_-\} = -i(D + iv_{01})$ , and  $\{\overline{Q}_+, D\} = -\partial_+\lambda_-$  one has

$$B = (D - iv_{01})\frac{\overline{\lambda}_-}{2}\left(1 - \frac{3}{2\pi|\sigma|^2}\right) + \frac{3i}{2\pi\sqrt{2}}\frac{\overline{\lambda}_-\overline{\lambda}_+\lambda_-}{\sigma^2\overline{\sigma}}. \quad (4.20)$$

To find the component  $b$  of  $B$  multiplying  $\overline{\lambda}_-$  for large  $\sigma$  is very easy: one replaces  $D - iv_{01}$  by its expectation value  $-it_R$ , and one can discard the terms proportional to  $1/|\sigma|^2$  or  $\overline{\lambda}_+\lambda_-$  because they are negligible for large  $\sigma$ . So one has simply  $b = -it_R/2$ . In computing

$\langle b\sigma^3 \rangle'$  we also need the behavior of the effective action in the large  $\sigma$  region: as derived above, this is

$$\exp(-S_{eff}) \sim \sigma^{-4} \exp\left(-\frac{Ae^2|t_R|^2}{2}\right). \quad (4.21)$$

The factor of  $\sigma^{-4}$  is from the “zero modes” analyzed above, while the factor of  $\exp(-\frac{Ae^2|t_R|^2}{2})$  comes from the behavior of the effective potential in the large  $\sigma$  region including the effects of the  $D$  term and the  $\theta$  angle as explained above.

Putting this together, we get

$$\begin{aligned} \langle b\sigma\sigma\sigma \rangle' &= \int_{\partial U} \frac{d\sigma}{\sigma} \frac{1}{2} (-it_R) \exp\left(\frac{-Ae^2|t_R|^2}{2}\right) \\ &= \pi t_R \exp\left(-\frac{Ae^2|t_R|^2}{2}\right) \end{aligned} \quad (4.22)$$

So the surface term is not zero and the naive conformal invariance is violated.

On  $\mathbf{S}^2$ , any two metrics are conformally equivalent, so the metric dependence of the correlation function can be determined by integrating the conformal anomaly. As the anomaly formula (4.22) depends only on the area and not on other details of the metric, the correlation function has the same property.

#### 4.2. Hamiltonian Formalism

Because of this anomaly in conformal invariance of the twisted model, we must proceed with care to extract the correct physics. We cannot simply compute a correlation function on  $\mathbf{S}^2$  with any given finite area: the result will depend on the area. It is clear, though, what we need to do to get the right result. If it is true that the linear sigma model flows in the infrared to the conformal field theory that we really want, then by taking  $A \rightarrow \infty$  in the linear model, we will get the desired results.

The precise way of taking  $A \rightarrow \infty$  does not matter, since as we have just seen the result only depends on the total  $A$ . The most obvious way to go to large  $A$  is to scale up the metric  $g$  of the surface by  $g \rightarrow e^\alpha g$  with  $\alpha$  a large constant. But in that limit, the linear model is not tractable: trying to compute in that region is like trying to explicitly understand the renormalization group flow of the model in the infrared.

An alternative way to go to large  $A$  is to expand the sphere in only one direction, so that it becomes a very long cigar, of circumference  $2\pi$  and length  $A/2\pi$ . We insert one copy of  $\sigma$  at the left end of the cigar, one at the right end, and one in the middle. In this limit we reduce to a Hamiltonian framework: the path integral on half a cigar with

$\sigma$  inserted at the tip gives a quantum state  $\Psi$ , and the correlation function we want is  $\langle \Psi | \sigma | \Psi \rangle$ . We can assume  $\Psi$  is a state of zero energy, since in the limit that the cigar is very long, other components are exponentially damped.

To find possible singularities, we only need to know how  $\Psi$  behaves for large  $\sigma$ . A zero energy state must obey

$$\overline{Q}_+ \Psi = Q_+ \Psi = 0. \quad (4.23)$$

This together with the quantum numbers determines the large  $\sigma$  behavior of  $\Psi$  up to a normalization constant. To find singularities, we only need to know the normalization constant up to a finite factor. This is given as follows. In computing  $\Psi(\sigma, \overline{\sigma}, \overline{\lambda}_-, \lambda_+)$ , one performs a path integral on a half-cigar with the boundary values fixed. Fixing the boundary values eliminates the zero modes, so there is no singularity in the wave-function;  $\Psi(\sigma, \dots)$  has a pointwise limit even when the bare couplings are taken to a value at which the theory becomes singular. Singularities in the correlation functions will have to arise because giving  $\Psi(\sigma, \dots)$  a limit as couplings approach a dangerous value with  $\sigma$  fixed will result in  $\Psi$  being unnormalizable (or so weakly normalizable that correlation functions diverge).

#### 4.3. The $\overline{\mathbf{27}}^3$ Coupling

In the large  $\sigma$  regime, we can set the  $\Phi^I = 0$  since their masses are of order  $\sigma$ . In this regime the model can be analyzed semiclassically (as done in some detail above) and we are free to work in the physical model where the fields have canonical spins. Taking this option, we have the following explicit expressions for the supercharges:

$$\begin{aligned} \overline{Q}_+ &= \int dx^1 (\sqrt{2} \lambda_+ \partial_+ \sigma - \lambda_- t_R) \\ Q_+ &= \int dx^1 (\sqrt{2} \overline{\lambda}_+ \partial_+ \overline{\sigma} - \overline{\lambda}_- \overline{t}_R) \end{aligned} \quad (4.24)$$

Truncating to zero modes and canonically quantizing, this becomes

$$\begin{aligned} \overline{Q}_+ &= \sqrt{2} \lambda_+ i \frac{\partial}{\partial \overline{\sigma}} - \lambda_- t_R \\ Q_+ &= \sqrt{2} \overline{\lambda}_+ i \frac{\partial}{\partial \sigma} - \overline{\lambda}_- \overline{t}_R \end{aligned} \quad (4.25)$$

The quantum wavefunction will depend on half the fermion variables; we take  $\Psi = \Psi(\sigma, \bar{\sigma}, \bar{\lambda}_-, \lambda_+)$ . The canonically conjugate fermions act by differentiation, as follows from the canonical anticommutation relations

$$\begin{aligned}\{\bar{\lambda}_-, \lambda_-\} &= 1, \quad \{\bar{\lambda}_+, \lambda_+\} = 1; \\ \{\lambda_-, \lambda_-\} &= 0 = \{\bar{\lambda}_-, \bar{\lambda}_-\}; \\ \{\lambda_+, \lambda_+\} &= 0 = \{\bar{\lambda}_+, \bar{\lambda}_+\}.\end{aligned}\tag{4.26}$$

We will look for a bosonic ground state wavefunction, killed by (4.25), of the form

$$\Psi(\sigma, \bar{\sigma}, \bar{\lambda}_-, \lambda_+) = f_1(\sigma, \bar{\sigma}) + f_2(\sigma, \bar{\sigma})\lambda_+\bar{\lambda}_-\tag{4.27}$$

Then the amplitude is

$$\int d^2\sigma d\lambda_+ d\bar{\lambda}_- \Psi \Psi \sigma = \int d^2\sigma f_1(\sigma, \bar{\sigma}) \sigma f_2(\sigma, \bar{\sigma})\tag{4.28}$$

Imposing (4.23) we find

$$\begin{aligned}\bar{Q}_+ \Psi &= 0 \Rightarrow i\sqrt{2}\bar{\partial}f_1 + t_R f_2 = 0 \\ Q_+ \Psi &= 0 \Rightarrow i\sqrt{2}\partial f_2 - \bar{t}_R f_1 = 0\end{aligned}\tag{4.29}$$

Combining these equations gives

$$(\partial\bar{\partial} - \frac{|t_R|^2}{2})f_1 = 0\tag{4.30}$$

For each partial wave sector, this equation has two solutions; for generic  $t_R$ , one of these vanishes exponentially at infinity and so is normalizable. (It is not obvious from this approximation that the solution that is normalizable for large  $\sigma$  is also regular near  $\sigma = 0$ , but that follows, for instance, from the representation of  $\Psi$  by a path integral on the half-cigar.) A singularity in  $\langle \sigma^3 \rangle$  can only come when the “regular” solution loses its normalizability, and this will only be for  $t_R = 0$ . (That  $t_R = 0$  is the precise location of the singularity can also be checked in other ways [27].)

It remains to determine whether there really is a singularity at  $t_R = 0$  and to compute its nature. Separating variables, we can write the solution in the form  $f_1 = N(t_R, \bar{t}_R)(\frac{\bar{\sigma}}{\sigma})^\nu \tilde{f}_1(|\sigma||t_R|)$  where  $N$  will be determined by the normalization conditions.

As explained above, the normalization condition is that for fixed  $\sigma$ , as  $t_R \rightarrow 0$ ,  $f_1$  (and  $f_2$ ) must be regular. At  $t_R = 0$ , (4.29) tells us that  $\bar{\partial}f_1 = 0$ . This combined

with the requirement of  $U(1)$  charge conservation in (4.28) implies  $f_1(\sigma, t_R = 0) = \frac{1}{\sigma} = (\frac{\bar{\sigma}}{\sigma})^{\frac{1}{2}} \frac{1}{|\sigma||t_R|} N(t_R)$  from which we learn that  $N(t_R) = |t_R|$  for small  $t_R$ . Similarly for  $t_R = 0$  we find  $\partial f_2 = 0$ . Then since  $\lambda_+ \bar{\lambda}_-$  has the same left and right  $U(1)$  charges  $(1, -1)$  as  $f_1$ , we learn that  $f_2$  is constant for small  $t_R$ .

Now that  $N$  is known, we can determine the behavior for large  $\sigma$  with fixed  $t_R$ . The solution of the radial equation is

$$f_1 = |t_R| \left(\frac{\bar{\sigma}}{\sigma}\right)^{\frac{1}{2}} \frac{e^{-\sqrt{2}|\sigma||t_R|}}{|\sigma|^{\frac{1}{2}}|t_R|^{\frac{1}{2}}} + \dots \quad (4.31)$$

where we have inserted the normalization factor  $N(t_R) = |t_R|$  and where  $\dots$  refers to the subleading terms in the asymptotic expansion of the solution. (The solution is in fact the Bessel function  $K_\nu(\sqrt{2}|\sigma||t_R|)$  with  $\nu = \frac{1}{2}$ ). From (4.29) we now find

$$f_2 = i \frac{\sqrt{2}}{t_R} \bar{\partial} f_1 = -i \frac{|t_R|}{t_R} \left[ \frac{1}{2\sqrt{2}|\sigma|} - |t_R| \right] \frac{e^{-\sqrt{2}|\sigma||t_R|}}{|\sigma|^{\frac{1}{2}}|t_R|^{\frac{1}{2}}} \quad (4.32)$$

So the amplitude becomes

$$\int d^2\sigma \sigma f_1 f_2 = \int d^2\sigma \sigma |t_R| \left(\frac{\bar{\sigma}}{\sigma}\right)^{\frac{1}{2}} \frac{e^{-2\sqrt{2}|\sigma||t_R|}}{|t_R||\sigma|} \frac{i|t_R|}{t_R} \left( \frac{1}{2|\sigma|} + |t_R| \right) \sim \frac{1}{t_R} \quad (4.33)$$

This agrees with the simple pole in the  $\overline{\mathbf{27}}^3$  coupling discovered using mirror symmetry by Candelas, de la Ossa, Green, and Parkes [28].

#### 4.4. $\mathbf{S}^3$ Couplings

To compute the dependence on gauge-singlet fields of the space-time superpotential  $W$ , we study three-point functions of the singlets. This allows us to read off three covariant derivatives of  $W$ . This follows from the fact that there is in the effective supergravity action a term  $\frac{1}{2}e^{K/2}D_i D_j W \chi^i \chi^j$ , where the  $\chi$ 's are matter fermions in spacetime and  $K$  is the spacetime Kahler potential. We will show that this coupling is zero by showing that keeping fixed all variables except the Kahler modulus  $t$ , the  $\mathbf{S}^3$  coupling has no pole.

The only possible pole would be at  $t_R = 0$ , where the model is singular; we must show that the  $\mathbf{S}^3$  couplings have no pole there.<sup>5</sup>

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<sup>5</sup> The  $\mathbf{S}^3$  couplings vanish for  $t = \infty$  because in this model the singlets are true moduli in the large radius, field theory limit. Of course, a holomorphic function on a compact complex manifold with a zero and no pole is identically zero. As noted in the introduction, because the superpotential is a section of a line bundle of negative curvature, its vanishing follows from absence of poles even if one does not know of a zero.

In this computation, it is simplest to use the half-twisted model, in which the spins of the fields are shifted by  $-J_R/2$  (plus possibly a shift proportional to the gauge charge  $Q$  chosen for convenience). This leads to the following spins for the worldsheet fields (final spins of zero are written in bold-face as those correspond to zero modes that are particularly important in what follows):

**Table 2: Spins in the half-twisted model**

<u>Fields</u>	<u><math>-(\frac{J_R}{2} - \frac{Q}{10})</math></u>	<u>Original Spin</u>	<u>Final Spin</u>
$\psi_+^i, \bar{\psi}_+^i$	(1/2, -1/2)	(1/2, 1/2)	(1, <b>0</b> )
$\bar{\psi}_-^i, \psi_-^i$	(0, 0)	(-1/2, -1/2)	(-1/2, -1/2)
$\psi_+^0, \bar{\psi}_+^0$	(0, 0)	(1/2, 1/2)	(1/2, 1/2)
$\bar{\psi}_-^0, \psi_-^0$	(1/2, -1/2)	(-1/2, -1/2)	( <b>0</b> , -1)
$\lambda_-, \bar{\lambda}_-$	(-1/2, 1/2)	(-1/2, -1/2)	(-1, <b>0</b> )
$\bar{\lambda}_+, \lambda_+$	(0, 0)	(1/2, 1/2)	(1/2, 1/2)
$\sigma, \bar{\sigma}$	(-1/2, 1/2)	(0, 0)	(-1/2, 1/2)
$s^i, \bar{s}^i$	(0, 0)	(0, 0)	( <b>0</b> , <b>0</b> )
$p, \bar{p}$	(-1/2, 1/2)	(0, 0)	(-1/2, 1/2)

With this prescription we obtain the gauge-invariant zero-mode integration measure

$$d\mu_h = d\bar{\lambda}_- d\bar{\psi}_-^0 \prod_i d\bar{\psi}_+^i \prod_i d^2 s^i \quad (4.34)$$

which has left and right  $U(1)$  charges  $(0, -3)$ . The  $\mathbf{S}^3$  correlation functions contain three vertex operators of charge  $(0,1)$  ( $q_L=0$  because these are gauge-singlets and  $q_R=1$  for bosonic vertex operators). So in the half-twisted model we can compute  $\mathbf{S}^3$  correlators with no extra spectral flow insertions required.

The vertex operators for singlets can be identified by their  $U(1)$  charges  $(q_L, q_R) = (0, 1)$ , their correspondence with parameters determining the size and shape of the manifold and vector bundle, and their relation to the  $(2,2)$  moduli on the  $(2,2)$  locus. The spacetime mode  $R$  corresponding to the Kahler parameter  $t$  comes from the worldsheet gauge multiplet. Its vertex operator must be annihilated by  $\bar{Q}_+$  but should be related by  $Q_+$  to the combination  $(D - iv_{01})$  which occurs in the worldsheet action with coefficient  $t$ . On the  $(2,2)$  locus it is also related by a *left*-moving supersymmetry generator to the

vertex operator for the  $\mathbf{10}_{-1}$  component of the  $\overline{\mathbf{27}}$ , which we determined above to be  $\sigma$ . These properties uniquely fix the vertex operator for  $R$ :

$$V_B^R = \lambda_- \quad (4.35)$$

The spacetime modes corresponding to the complex structure and bundle parameters should be associated with sets of five quartic polynomials  $H_i^{(4)}(s^j)$  subject to one quintic relation. They must be annihilated by  $\overline{Q}_+$ . On the  $(2,2)$  locus the ones corresponding to complex structure deformations must be related by a left-moving supersymmetry generator to the vertex operator for the  $\mathbf{10}_1$  component of the  $\mathbf{27}$  vertex operator. The latter are given by quintic polynomials in the  $s^j$  (these have left and right  $U(1)$  charges  $(1,1)$  and are annihilated by  $\overline{Q}_+$ ):

$$V_B^{\mathbf{10}_1} = pL^{(5)}(s^j) \quad (4.36)$$

We have normalized this vertex operator to be independent of  $r$ , ensuring that the  $\mathbf{27}^3$  coupling is constant on the Kahler moduli space on the  $(2,2)$  locus [8].

Now let us discuss the vertex operators for  $E_6$  singlets coming from complex structure and bundle deformations. Consider the vertex operator

$$V_S = 5pH_i^{(4)}(s^j)\psi_-^i + L^{(5)}(s^i)\psi_-^0 \quad (4.37)$$

where  $H^{(4)}$  and  $L^{(5)}$  are quartic and quintic polynomials, respectively. The condition for the operator to be  $\overline{Q}_+$ -invariant is  $s^i H_i^{(4)} = L^{(5)}$ . 101 such operators obey  $H_i^{(4)} = \frac{\partial L^{(5)}}{\partial s^i}$  and are related near field theory to the deformation of the complex structure of the manifold. On the  $(2,2)$  locus, these operators are produced by acting on the operators in (4.36) with  $Q_-$ . We normalized the vertex operator (4.36) to be independent of  $r$ . Then the normalization of the complex structure vertex operators discussed here follows from their relation to the  $\mathbf{27}$  on the  $(2,2)$  locus. In any case it is clear that the vertex operators for the complex structure and bundle deformations should be independent of  $r$  from the fact that the terms in the linear sigma model action involving the parameters  $J_{i,j_1 j_2 j_3 j_4}$  are decoupled from the term proportional to  $r$ .

In the  $\overline{\mathbf{27}}^3$  coupling computed in the last subsection, we found a simple pole singularity for  $t \rightarrow 0$  due to the unbounded zero-mode of  $\sigma$ . From the above table of half-twisted spins, we immediately see that no singularity is possible for the  $\mathbf{S}^3$  correlator, for the simple reason that  $\sigma$  has acquired a spin! There is now no bosonic zero-mode (recall that in this regime the  $s_i$  have huge masses of order  $\sigma$ ) and hence no pole. As discussed in section 1, the spacetime superpotential must diverge on a locus of codimension one in the moduli space if it is not to vanish identically. We see here that this cannot happen, and that therefore the superpotential is flat and the singlets are good moduli.

#### 4.5. $\overline{\mathbf{27}}^3$ Revisited

This argument may seem a little too quick, for the following reasons:

(1) We obtained a simple pole for the the  $\mathbf{16} - \mathbf{16} - \mathbf{10}$  component of the  $\overline{\mathbf{27}}^3$  coupling by doing the computation in the physical model in section 4.3. We then argued that the  $\mathbf{S}^3$  correlator is naturally done in the half-twisted model and thus cannot have a similar pole because  $\sigma$  is twisted. How do we reproduce the pole in the  $\mathbf{16} - \mathbf{16} - \mathbf{10}$  coupling in the half-twisted model?

(2) By  $E_6$  symmetry we should be obtain the same simple pole by computing the  $\mathbf{10} - \mathbf{10} - \mathbf{1}$  component of the  $\overline{\mathbf{27}}^3$  coupling.

We will explain presently the resolution to problem (1). There the spin operator insertions needed to perform the  $\mathbf{16} - \mathbf{16} - \mathbf{10}$  computation in the half-twisted model make up for the twisting of  $\sigma$  and we can recover the simple pole. As for case (2), we find ourselves unable to find a natural representative for the vertex operator for the  $\mathbf{1_2}$  mode in the  $\mathbf{10} - \mathbf{10} - \mathbf{1}$  computation. (This may be related to the fact that this mode appears in a twisted sector at the Landau-Ginzburg point. It is certainly related to the fact that the  $(0, 2)$   $E_6$  models that we are studying do not have deformations to  $(0, 2)$  linear sigma models with  $SO(10)$  gauge group. <sup>6</sup>) This leaves us without a satisfying answer to question (2) within the linear sigma model. However, we have no such difficulties with the singlet coupling  $\mathbf{S}^3$ . (The singlet vertex operators and their normalizations are conveniently determined in the linear sigma model as described in the previous section.) Therefore we remain convinced of the absence of a pole in the singlet coupling and the flatness of the spacetime superpotential.

To answer the first question, we will translate the above computation of the  $\mathbf{16-16-10}$  amplitude into a half-twisted computation, working with  $\sigma$  and  $\lambda$  twisted as in table 2. But now we must explicitly insert the left spectral-flow generators (i.e. internal part of the gaugino vertex operator for the  $U(1)$  which combines with  $SO(10)$  to form a maximal subgroup of  $E_6$ ) to put the fermion states in the  $\overline{\mathbf{16}}_{1/2}$  (spinor) representation of  $SO(10)$ . It is not clear how to represent these insertions by linear sigma model operators, but luckily there is another option: exponentiate them, explicitly adding to the action the extra couplings of the fields to the spin connection as described in section 3.

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<sup>6</sup> That in fact is why, in order to study the  $SO(10)$  models, Distler and Kachru modified the model to eliminate the  $\sigma$  field [10]; for reasons that are still not entirely clear, this did not give a model with the right properties.



In the half-twisted model we would compute

$$< \left( \oint_{C_{z_1}} dw \Sigma(w) V_B^{\mathbf{10}-1}(z_1, \bar{z}_1) \right) V_B^{\mathbf{10}-1}(z_2, \bar{z}_2) \left( \oint_{C_{z_3}} dv \Sigma(v) V_B^{\mathbf{10}-1}(z_3, \bar{z}_3) \right) >_{half} \quad (4.38)$$

where  $\Sigma$  is the left-moving analogue of the holomorphic part of the spacetime supersymmetry generator; in the CFT its internal part would be given by  $e^{i\frac{\sqrt{3}}{2}H}$  where  $H$  is the bosonization of the left-moving  $U(1)$  current. In the CFT we could achieve these insertions by adding to the action a term proportional to  $\int H \hat{R}$  where  $\hat{R}$  is the worldsheet curvature which has delta-function support on the insertions. Here we are working on an infinitely elongated sphere, so that the bulk of the worldsheet is a flat cylinder and the curvature is pushed out to the ends at infinity. So we will take a step function for the spin connection, which is constant everywhere except at the insertions. On the bulk of the worldsheet, which is a flat cylinder, this constant spin connection contribution shifts the boundary conditions of the fields around the cylinder since  $\oint \omega_1 dx^1 = 1$ . The resulting contribution to the curvature  $\hat{R}$  is  $\sim \delta(\tau - T)$ , localizing the insertion on a contour  $\tau = T$  surrounding the operator on the end of the cylinder.

To implement this plan in the Hamiltonian framework we must use the following twisted mode expansions, which follow from the spins in the half-twisted model (table 2):

$$\begin{aligned} \sigma &= a(\tau) e^{i\frac{x}{2}} + b(\tau) e^{-i\frac{x}{2}} + \dots \\ \bar{\sigma} &= \bar{a}(\tau) e^{-i\frac{x}{2}} + \bar{b}(\tau) e^{i\frac{x}{2}} + \dots \end{aligned} \quad (4.39)$$

and

$$\begin{aligned} \lambda_+ &= \tilde{\alpha}(\tau) e^{i\frac{x}{2}} + \tilde{\rho}(\tau) e^{-i\frac{x}{2}} + \dots \\ \bar{\lambda}_+ &= \alpha(\tau) e^{-i\frac{x}{2}} + \rho(\tau) e^{i\frac{x}{2}} + \dots \\ \lambda_- &= \gamma(\tau) e^{-ix} + \xi(\tau) + \dots \\ \bar{\lambda}_- &= \bar{\lambda}_-(\tau) + \dots \end{aligned} \quad (4.40)$$

Our states will be killed by the supersymmetry generators

$$\begin{aligned} \bar{Q}_+ &= \int dx \left( \sqrt{2} \lambda_+ (\partial_+ - \frac{i}{2} \omega_+) \sigma - t \lambda_- \right) \\ Q_+ &= \int dx \left( \sqrt{2} \bar{\lambda}_+ (\partial_+ + \frac{i}{2} \omega_+) \bar{\sigma} - \bar{t} \bar{\lambda}_- \right) \end{aligned} \quad (4.41)$$

Note that to implement the insertion of the left spectral flow operators, we have explicitly included the spin connection terms. This results in the shift  $\partial_+ \rightarrow \partial_+ + i\omega_+ \frac{q_L}{2}$  where  $\omega_+$

is the  $+$  component of the spin connection. For the first component of the spin connection we take

$$\omega_1 = \lim_{T \rightarrow \infty} \begin{cases} \theta(\tau + T), \tau < T; \\ \theta(T - \tau), \tau > -T. \end{cases} \quad (4.42)$$

where  $\theta(y)$  is the standard step function. ( $\theta(y) = 1$  for  $y \geq 0$ ;  $\theta(y) = 0$  for  $y < 0$ .) In terms of canonical variables (4.41) becomes

$$\begin{aligned} \bar{Q}_+ &= \sqrt{2}\tilde{\rho}\left(i\frac{\partial}{\partial\bar{a}} + \frac{ia}{2}(1 - \omega_1)\right) + \sqrt{2}\tilde{\alpha}\left(i\frac{\partial}{\partial\bar{b}} - \frac{ib}{2}(1 + \omega_1)\right) - t\xi \\ Q_+ &= \sqrt{2}\rho\left(i\frac{\partial}{\partial a} - \frac{i\bar{a}}{2}(1 - \omega_1)\right) + \sqrt{2}\alpha\left(i\frac{\partial}{\partial b} + \frac{i\bar{b}}{2}(1 + \omega_1)\right) - \bar{t}\bar{\lambda}_- \end{aligned} \quad (4.43)$$

With the choice (4.42) of spin connection we effectively have a “sudden” perturbation turned on at  $\tau = \pm T$ . The Hamiltonian for  $|\tau| > |T|$  has a quadratic potential for the modes  $a$  and  $b$ , but in the bulk ( $|\tau| < |T|$ ) we have a constant  $\omega_1 = 1$  so the potential for  $a$  is turned off: a zero-mode for  $a$  has arisen because of the coupling to the spin connection (said differently,  $\sigma$  is not constant but is effectively covariantly constant, which is what counts in computing the energy once the spin connection coupling is included). So we can reproduce the pole in the **16** – **16** – **10** coupling in the half twisted model.

## 5. Behavior for singular $F_{i,j_1j_2j_3j_4}$

### 5.1. Linear Sigma Model Analysis

We now discuss what happens at the other type of singularity discussed in section 2, where the complex structure of the manifold and/or the gauge bundle degenerates, keeping fixed the Kahler parameter  $t = ir + \theta/2\pi$ .

Since the part of the bosonic potential that comes from the superpotential is

$$|p|^2 \sum_i |\tilde{J}_i|^2 + |G|^2, \quad (5.1)$$

for the charged fields to be able to go to infinity with finite cost in energy requires that there be a non-trivial solution of

$$\tilde{J}_i = G = 0. \quad (5.2)$$

(The  $D$  terms are such for some  $s^i$  to go to infinity at finite cost in energy,  $p$  must also, and vice-versa.) As  $5G = \sum_i s^i \tilde{J}_i$ , there are only five independent equations in (5.2); on the

other hand, because the  $\tilde{J}_i$  are homogeneous functions of the  $s^i$ , in looking for a non-trivial function there are only four independent variables, the ratios  $s^i/s^5$  with  $i \leq 4$ . Hence (5.2) generically has no solutions. In looking for a pole in the space-time superpotential, we need to consider the behavior when one parameter is varied in the  $\tilde{J}_i$  – as poles, if they exist, arise in codimension one on the parameter space. By varying this one parameter in the  $\tilde{J}_i$  together with the four  $s^i/s^5$ , we have five parameters in all, so generically we can expect finitely many isolated solutions to the five equations  $\tilde{J}_i = 0$ .

This means that the situation that we need to consider is that where, for an exceptional value of the parameters, there is on  $\mathbf{CP}^4$  an isolated solution of  $\tilde{J}_i = 0$ . This singularity will occur on a generic point  $w$  on  $\mathbf{CP}^4$  – to impose a restriction on where on  $\mathbf{CP}^4$  the solution will arise, one would need to adjust more than one parameter in the  $\tilde{J}_i$ . We may as well assume that the solution is at  $s^i = (0, 0, 0, 0, 1)$ . Also, when just one parameter in the  $\tilde{J}_i$  is adjusted, the Jacobian  $\det(\partial_i \tilde{J}_j)$  will be generically non-zero at  $w$ .

As practice, let us first show that the  $\mathbf{27}^3$  Yukawa couplings have a pole in this situation (on the  $(2, 2)$  locus this is a standard result due to [29] and [30].) In the generic situation described above, the only charged fields that can become large are  $s_5$  and  $p$ . They must become large in proportion because of the  $D$  terms, and in the region  $s^5 \rightarrow \infty$ ,  $\sigma$  is very massive. We therefore get an effective computation on the  $s^5$  plane which turns out to be very similar to the computation done earlier on the  $\sigma$  plane for the  $\overline{\mathbf{27}}^3$  case.

We consider a family of models, parametrized by a complex parameter  $\epsilon$ , such that at  $\epsilon = 0$ , the potential vanishes for  $s^i = (0, 0, 0, 0, s^5)$  and  $p = cs^5$ . For small, non-zero  $\epsilon$ ,  $s^1, \dots, s^4$  will still be hugely massive in the regime of large  $s^5$ . For  $s^i = (0, 0, 0, 0, s^5)$  and  $p = cs^5$ , the only relevant contributions to the  $J_i$  are those proportional to  $(s^5)^5$ ; moreover, these vanish if  $\epsilon = 0$  and in general are proportional to  $\epsilon$ . So in this region we have  $J_i = p\tilde{J}_i = \epsilon K_i (s^5)^5$  for some constants  $K_i$ , and  $J_0 = \epsilon K_0 (s^5)^5$  for some constant  $K_0$ . With these simplifications near the singular locus, we will find that the computation of the  $\mathbf{27}^3$  coupling in this regime becomes isomorphic to that of the  $\overline{\mathbf{27}}^3$  coupling evaluated near  $t=0$  in section 3. Recall that we reduced that computation to a Hamiltonian computation involving wavefunctions  $\Psi$  annihilated by  $\overline{Q}_+$  and  $Q_+$ . In the regime of interest here (large  $s^5$  and  $p$ ) the  $\sigma$  and gauge multiplets are very heavy and can be set to zero. Similarly we only have one bosonic zero mode  $s^5$  and the  $s^1, \dots, s^4$  multiplets are massive. This leaves the following expressions for  $\overline{Q}_+$  and  $Q_+$ :

$$\begin{aligned}
\overline{Q}_+ &= -\sqrt{2}\overline{\psi}_+^I \frac{\partial}{\partial \overline{\phi}^I} + J_I \psi_-^I \equiv -\sqrt{2}\overline{\alpha}_+ \frac{\partial}{\partial \overline{s}^5} + \epsilon(s^5)^5 \alpha_- \\
Q_+ &= \sqrt{2}\psi_+^I \frac{\partial}{\partial \phi^I} + \overline{J}_I \overline{\psi}_-^I \equiv \sqrt{2}\alpha_+ \frac{\partial}{\partial s^5} + \overline{\epsilon}(\overline{s}^5)^5 \overline{\alpha}_-
\end{aligned}
\tag{5.3}$$

where  $\alpha_+ = \psi_+^0/c + \psi_+^5$  and  $\alpha_- = K_I \psi_-^I$  (and similarly for the complex conjugates). This only depends on  $s^5$  as it should.

To make the isomorphism with the  $\mathbf{27}^3$  computation manifest, we change variables from  $s^5$  to  $U(s^5) = (s^5)^6/6$  and from  $\alpha_{\pm}$  to  $\chi_- = -\alpha_- \frac{\partial U}{\partial s^5} = -\alpha_-(s^5)^5$  and  $i\chi_+ = \frac{\partial U}{\partial \overline{s}^5} \alpha_+ = (s^5)^5 \alpha_+$ . Then

$$\begin{aligned}
\overline{Q}_+ &= \sqrt{2}\overline{\chi}_+ i \frac{\partial}{\partial \overline{U}} - \epsilon \chi_- \\
Q_+ &= \sqrt{2}\chi_+ i \frac{\partial}{\partial U} - \overline{\epsilon} \overline{\chi}_-
\end{aligned}
\tag{5.4}$$

These equations are isomorphic to (4.25). We can now look for a wavefunction  $\Psi(s^5, \overline{s}^5, \overline{\chi}_-, \chi_+)$  which is annihilated by  $\overline{Q}_+$  and  $Q_+$ . So correlation functions of three  $\mathbf{27}$ 's (quintic polynomials) with a term proportional to  $(s^5)^5$  will have a simple

pole at the conifold singularity, a result derived on the (2,2) locus by Candelas at large radius [29], and by Vafa at LG [30].

Now let us consider our real interest: the  $\mathbf{S}^3$  couplings. Once again the simplest direct approach is to compute in the half-twisted model with spins shifted as in table 2. In particular  $p$  is twisted (and even if one adds a multiple of  $Q$  in the twisting, which could occur dynamically if the vertex operators are enveloped by a fractional instanton,  $p$  or  $s^5$  is twisted). Vertex operators for the singlets (killed by  $\overline{Q}_+$ ) were given in (4.35) and (4.37).

They have  $U(1)$  charges (0,1) and the  $\mathbf{S}^3$  coupling is given by a correlation function of three such vertex operators with no extra insertions of spectral flow generators. Again there can be no divergence, this time because of the twisting of  $p$ . We thus confirm at this singular locus what we learned at the other: the superpotential is flat.

## 5.2. Large Radius Analysis: Worldsheet Instantons

The rest of this paper has a somewhat different emphasis. We want to show that under rather certain assumptions, any pole in the superpotential of a (0,2) model can only arise by setting the complex structure or Kahler moduli to particular values; one cannot get a pole that arises upon adjusting the bundle moduli on a generic smooth Calabi-Yau with

generic metric. We hope that in future this will be useful in understanding the behavior of  $(0, 2)$  models.

The discussion will be carried out by looking at the sum over worldsheet instantons. If the superpotential has a singularity for generic values of Kahler and complex structure moduli (but some special bundle moduli) this singularity must come not from summing over the worldsheet instanton number; rather, a singularity must arise in the worldsheet instanton contributions for some specific values of the worldsheet instanton number. We will explore the conditions under which this might happen. These will be practically the only remarks in the present paper that are not limited to those models that are associated with linear sigma models.

In the large radius limit,  $(0, 2)$  models are defined by choosing (on some Calabi-Yau manifold  $X$ ) a solution of the Kahler-Yang-Mills equations

$$g^{i\bar{j}} F_{i\bar{j}} = 0 = F_{ij} = F_{\bar{i}\bar{j}}. \quad (5.5)$$

Here  $g$  is the Calabi-Yau metric and  $F$  is the Yang-Mills field strength. We would like to make some simple remarks about how solutions of this equation can develop singularities. These remarks are not limited to the case that  $X$  has  $c_1 = 0$ , though that is where we will apply them.

First of all, we consider the case that the complex dimension of  $X$  is one. Then (5.5) says that the gauge field is flat; in particular, it can be gauged away locally and cannot develop any singularities at all. This simple statement for  $\dim_{\mathbf{C}}(X) = 1$  is a special case of a more general statement: a family of solutions of (5.5) cannot develop a singularity in complex codimension one.

Now we move on to the case that the complex dimension of  $X$  is two. Then (5.5) is equivalent to the Yang-Mills instanton equations. In this case, it is very familiar that even for  $X = \mathbf{C}^2 \cong \mathbf{R}^4$ , a singularity can develop as an instanton shrinks to zero size. The singularity arises at an isolated point, which has complex codimension two. To achieve this singularity, in the usual rotation-invariant description of instantons on  $\mathbf{R}^4$ , one parameter, the instanton scale size, is adjusted. If a complex structure is chosen to identify  $\mathbf{R}^4$  as  $\mathbf{C}^2$ , the scale size becomes the absolute value of a complex variable. Thus, by adjusting one complex parameter in the parameter space of solutions of (5.5), one can produce a singularity of the solution that arises in complex codimension two on  $X$ .

More generally, for  $X$  of any complex dimension, in a one-parameter family of solutions of (5.5), one will generically meet singularities that will arise on  $X$  on any complex

codimension  $\geq 2$ . In the last paragraph, we noted a familiar example of a singularity in complex codimension two; the quintic, as we saw in section 5.1, gives a simple example where a singularity develops in complex codimension three.

Now let us consider a – fallacious – attempt to prove that in any  $(0, 2)$  models, the contribution of a given worldsheet instanton can never develop a singularity as the bundle parameters are varied for generic values of the complex structure and Kahler moduli of  $X$ .

Call the instanton  $C$ . It is a holomorphic curve in  $X$  and is of genus zero. We will suppose that  $C$  is isolated. (For example, for generic complex structure on the quintic, the worldsheet instantons are all isolated.) For the contribution of  $C$  to develop a pole as the gauge bundle is varied, keeping the complex structure of  $X$  fixed, the gauge bundle must develop a singularity on  $C$ . Otherwise, the evaluation of the contribution of the given instanton is manifestly finite.

Now, let  $n$  be the complex dimension of  $X$ .  $C$  has complex dimension 1 or codimension  $n-1$ . From the general discussion of solutions of (5.5), as a parameter is varied singularities will develop on a submanifold  $Y$  of codimension  $\geq 2$ . Since  $(n-1) + 2 > n$ , it would appear that generically we should expect the intersection of  $C$  and  $Y$  to be empty. Then the contribution of  $C$  cannot generate a pole as it does not “see” the singularity.

What is wrong with this argument? The only fallacy is that  $Y$  may not be sufficiently generic to allow such simple dimension counting.

Let us spell this out more precisely when  $X$  has complex dimension three. Then  $Y$  has dimension one or zero; consider first the case that the dimension is one. With  $C$  and  $Y$  both of dimension one, and  $1 + 1 < 3$ , we would not expect  $C$  and  $Y$  to meet. In fact, dimension-counting suggests (though it is perhaps hard to prove) that on most Calabi-Yau manifolds it is possible to pick a complex structure such that distinct curves  $C$  and  $Y$  never meet. If this is so, how can the contribution of  $C$  to the superpotential get a pole when the bundle degenerates on  $Y$ ? The answer – pointed out to us by D. Morrison – is that such a pole can arise precisely when  $Y = C$ ! Thus, we get a partial criterion for failure of conformal invariance of  $(0, 2)$  models: the space-time superpotential should be expected to develop a pole, and thus conformal invariance should be expected to fail, if the gauge bundle can degenerate on a curve of genus zero.

While curves on Calabi-Yau three-folds are generically isolated, points are always free to move. With this in mind, consider the other possibility, that  $Y$  is a point, of dimension zero. In this case, consider a generic one-parameter family of gauge bundles with singularities developing on isolated points. If all is sufficiently “generic,” one would

expect that in a generic degeneration,  $Y$  would not be located on  $C$  and therefore the contribution to the superpotential from  $C$  could not diverge. However, complex geometry sometimes plays tricks, and for all we know there may be Calabi-Yau manifolds and gauge bundles such that the singularities that arise in one-parameter families always land on rational curves.

Anyway, we get our criterion for conformal invariance of a  $(0, 2)$  model defined on a Calabi-Yau threefold: if the singularities in a generic one-parameter family of gauge bundles lie either on curves of genus  $\geq 1$  that do not meet curves of genus zero, or on points that generically do not lie on curves of genus zero, then the space-time superpotential cannot have a pole for generic values of the Kahler and complex structure moduli. (At special Kahler moduli, there may be a pole from summing over instantons, and at special complex structure moduli, a pole may come if  $C$  is contained in, or at least intersects, singularities of  $X$ .)

This criterion is easy to implement for the quintic. As we saw in section 5.1, in a generic family, the gauge bundle degenerates only on isolated points, which moreover can move freely. So our criterion is obeyed. It seems very likely that the same reasoning will work for a large class of  $(0, 2)$  models derived from linear sigma models, but we will not analyze this here.

### *More Detail For The Quintic*

Let us describe this in more detail. Consider a generic one parameter family of quintic models leading (as in section 5.1) to an isolated singularity which we may as well take to lie at  $x = (0, 0, 0, 0, 1)$ . This means that the five functions  $\tilde{J}_i$  all vanish at  $x$ . We want to show that for generic such  $\tilde{J}_i$ , there is no rational curve on  $X$  that passes through  $x$ . In general, a rational curve is a map from  $\mathbf{CP}^1$ , with homogeneous coordinates  $(u, v)$ , to  $X$ ; the homogeneous coordinates  $s^i$  of  $X$  are homogeneous functions of  $u, v$  of some degree  $k$ , called the degree of the instanton. We will for simplicity take  $k = 1$ , but the counting works similarly in general. There is no essential loss in assuming that a curve that passes through  $x$  does so at  $u = 0$ , and then the curve takes the form

$$s^i(u, v) = (\alpha_1 u, \alpha_2 u, \alpha_3 u, \alpha_4 u, v). \quad (5.6)$$

(We imposed that  $s^i(0, v) = (0, 0, 0, 0, v)$ , and added to  $v$  a multiple of  $u$  to ensure  $s^5 = v$ . The  $\alpha_i$  cannot be all zero or (5.6) would not define a curve at all.) (5.6) gives a linear map

from  $\mathbf{CP}^1$  to  $\mathbf{CP}^4$  that meets  $x$ ; we must determine if it lies on  $X$ . Setting  $G = s^i \tilde{J}_i$  (so that  $X$  is defined by  $G = 0$ ), the curve lies on  $X$  if and only if

$$G(s^i(u, v)) = 0. \quad (5.7)$$

Note that  $G(s^i(u, v))$  is a homogeneous function of  $u, v$  of degree five. The condition that  $\tilde{J}_i = 0$  at  $x$  means that  $G(s^i(0, v)) = 0$ , but  $G$  is otherwise generic. So we can expand

$$G(s^i(u, v)) = u^5 P_5(\alpha_i) + u^4 v P_4(\alpha_i) + \dots + uv^4 P_1(\alpha_i). \quad (5.8)$$

For  $G(s^i(u, v))$  to be identically zero is five equations for the four  $\alpha_i$ , so generically there is no solution. The essential point is that the condition that the curve passes through  $x$  eliminates two parameters that could otherwise be added to (5.6) (plus more that can be rotated away by linear transformations of  $u, v$ ), but the condition that  $\tilde{J}_i$  are all zero at  $x$  eliminates only one term from (5.8), namely the coefficient of  $v^5$ . So in asking that there is a singularity and that the curve passes through it, we have eliminated two variables and only one equation. Without these conditions, the curves on the quintic are generically isolated; with the conditions, they generically do not exist.

### 5.3. Localization of the Half-Twisted Path Integral

In the above subsection, we showed that generically, the instanton does not meet the possible solutions of the equations  $\tilde{J}_i = 0$  and thus the possible flat directions of the potential. Actually, one can also argue directly that nothing goes wrong to instanton computations in the linear sigma model even if an instanton does meet a solution of  $\tilde{J}_i = 0$ ; in fact, in the instanton sector we can even set  $\tilde{J}_i$  to be identically zero without producing a singularity. The reason is that the computation of instanton contributions localizes, as explained in [9], section 3.4, on the space of vortex solutions of an abelian Higgs model that also obey  $J_i = 0$ . For  $(0, 2)$  models the fixed point locus is determined by setting  $\{\bar{Q}_+, F\} \equiv 0$  for all fermion fields  $F$ . As long as the instanton number is non-zero, either  $p$  or all the  $s^i$  must vanish in the instanton solution because of having the wrong sign of the charge.

Once that is given, the space of vortex solutions remains compact no matter how the  $\tilde{J}_i$  degenerate – even if one takes  $\tilde{J}_i$  to zero. (This is somewhat analogous to the fact that in the  $(2, 2)$  case, instanton computations can be performed at  $G = 0$  [27].) Hence the instanton calculation can never develop a pole for given instanton number.

Of course, the argument just given uses the structure of the linear sigma model while the earlier argument that singularities require that the gauge bundle should degenerate in complex codimension two is valid for arbitrary  $(0, 2)$  models constructed from a Calabi-Yau manifold with a holomorphic vector bundle.



## 6. Discussion

Although we concentrated on the quintic vacuum, it is clear that our arguments apply more generally. Any linear sigma model describing the parameter space of a complete intersection in a toric variety will have bosonic fields charged under the  $U(1)$   $R$ -symmetry in such a way that the argument of section 4.3 will go through. Similarly, at a generic point on the singular locus the vector bundle  $V$  will be singular but the manifold will be smooth.

This result is surprising from the point of view of worldsheet instantons: it can be taken as a prediction that the contributions to singlet correlations of the 2875 rational curves on the quintic cancel. At the Landau-Ginzburg end, our conclusion also implies that *all*  $E_6$  singlets that correspond to deformation-theoretic modes at large radius must have flat superpotential at small radius; this includes amplitudes which have no symmetry reason to vanish [11]. In [31], this has been verified for one such amplitude involving singlets at  $r = -\infty$  which appear to correspond to 24 of the 224  $H^1(\text{End } V)$  modes at large radius.

The next natural question is to consider giving gauge *non*-singlet scalars vacuum expectation values, breaking the gauge group down to  $SO(10)$  or  $SU(5)$ . Obtaining, for example, an  $SO(10)$  theory by turning on the  $\mathbf{1}_2$  component of the  $\overline{\mathbf{27}}$  and the  $\mathbf{1}_{-2}$  component of the  $\mathbf{27}$  does not simply involve adding polynomial deformations to the linear sigma model. However, Distler and Kachru have introduced linear sigma models describing  $(0, 2)$  models which are *not* deformations of  $(2, 2)$  models [10]. These have smaller gauge groups at the string scale. For these cases one must first check that the nonlinear sigma models to which they reduce at large radius are solutions of the low-energy equations of motion. This ensures that the superpotential does not diverge at infinite radius.

Then the methods used in this paper for the quintic can be invoked to argue for the conformal invariance of the infrared limit of these more realistic models. The twisting of the fields by their  $U(1)_R$  charges ensures that the singularities of the linear sigma model do not lead to poles in singlet correlation functions. Similarly, the worldsheet instanton computations will not “know” about the singular locus of the vector bundle. It should now be clear how to apply our considerations to still more general linear sigma models (such as models with no large radius phase at all [32]): (i) check that the fields that could potentially go to infinity when the model becomes singular have nonzero  $U(1)_R$  charges, or (ii) study the instanton expansion about a known locus on the parameter space and check whether the instanton moduli space is compact and thus avoids the singularity.

In addition to fixing the space-time superpotential for the singlets, the methods used here might help compute other quantities of interest for  $(0, 2)$  vacua. In principle one can reliably compute quantities which are renormalization-group invariant and holomorphic up to surface terms in the linear sigma model. In practice this is difficult except in certain limits where the computation becomes semiclassical. As we have seen, one such limit is near the singularities. The orders of the poles in the Yukawa couplings of gauge-charged modes near the singularities can be computed for more models. Another quantity of great interest is the one-loop gauge coupling function, which bears on the unification scale in string theory and the moduli-dependence of non-perturbative effects. Determining its behavior near the singularities using the linear sigma model might go a long way toward fixing its dependence on the moduli (including  $(0, 2)$  moduli).

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## References

- [1] T. Banks, L. Dixon, D. Friedan, and E. Martinec, “Phenomenology and Conformal Field Theory, or Can string theory predict the weak mixing angle?”, *Nucl. Phys.* **B299** (1988) 613.
- [2] L. Dixon, “Some World-Sheet Properties of Superstring Compactifications, in Orbifolds and Otherwise”, Lectures given at the 1987 ICTP Summer Workshop in High Energy Physics and Cosmology.
- [3] X.G. Wen and E. Witten, “World Sheet Instantons and the Peccei-Quinn Symmetry”, *Phys. Lett.* **166B** (1986) 397.
- [4] M. Dine, N. Seiberg, X.G. Wen, and E. Witten, “Nonperturbative Effects on the String Worldsheet” I and II, *Nucl. Phys.* **B278** (1986) 769 and **B289** (1987) 319.
- [5] J. Distler, “Resurrecting (0,2) Models”, *Phys. Lett.* **B188** (1987) 431.
- [6] M. Cvetič, “Exact Construction of (0,2) Calabi-Yau Manifolds”, *Phys. Rev. Lett.* **59** (1987) 2829.
- [7] M. Dine and N. Seiberg, “Are (0,2) Models String Miracles?”, *Nucl. Phys.* **B306** (1988) 137.
- [8] J. Distler and B. Greene, “Aspects of (2,0) String Compactifications”, *Nucl. Phys.* **B304** (1988) 1.
- [9] E. Witten, “Phases of N=2 Theories in Two Dimensions”, *Nucl. Phys.* **B403** (1993) 159.
- [10] J. Distler and S. Kachru, “(0,2) Landau-Ginzburg Theory”, *Nucl. Phys.* **B413** (1994) 213, hep-th/9309110.
- [11] J. Distler and S. Kachru, “Singlet Couplings and (0,2) Models”, *Nucl. Phys.* **B430** (1994) 13, hep-th/9406090.
- [12] E. Witten and J. Bagger, “Quantization of Newton’s Constant in Certain Supergravity Theories”, *Phys. Lett.* **115B** (1982) 202.
- [13] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, 2nd ed., Princeton University Press, 1992.
- [14] T. Hubsch, *Calabi-Yau Manifolds: A Bestiary For Physicists* (World-Scientific, 1992).
- [15] N. Seiberg, “The Power Of Holomorphy: Exact Results in 4-D SUSY Field Theories”, RU-94-64 (1994) hep-th/9408013.
- [16] E. Witten, “Topological Sigma Models”, *Comm. Math Phys.* **118** (1988) 411.
- [17] T. Eguchi and S.K. Yang, “N=2 Superconformal Models as Topological Field Theories”, *Mod. Phys. Lett.* **A5** (1990) 1693.
- [18] E. Witten, “Mirror Manifolds and Topological Field Theory”, in *Essays on Mirror Manifolds*, ed. S-T Yau, International Press (1992) 120.
- [19] E. Witten, “On the Structure of the Topological Phase of Two-Dimensional Gravity”, *Nucl. Phys.* **B340** (1990) 281.

- [20] S. Cecotti and C. Vafa, “Topological- Anti-Topological Fusion” *Nucl. Phys.* **B367** (1991) 359.
- [21] Antoniadis, Gava, Narain, and Taylor, “Topological Amplitudes in String Theory”, *Nucl. Phys.* **B413** (1994) 162.
- [22] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, “Kodaira-Spencer Theory of Gravity and Exact Quantum String Amplitudes”, *Commun. Math. Phys.* **165** (1994) 311.
- [23] D. Gepner, “Lectures on N=2 String Theory”, Lectures given at the Spring School on Superstrings, Trieste, Italy, 3-11 April, 1989, PUPT-1121.
- [24] S. Coleman, “More on the Massive Schwinger Model”, *Ann. Phys.* **101** (1976) 239.
- [25] E. Witten, “The Verlinde Algebra and the Cohomology of the Grassmannian”, IASSNS-HEP-93/41, hep-th/9312104.
- [26] A. D’Adda, A.C. Davis, P. Di Vecchia, and P. Salomonson, “An Effective Action for the Supersymmetric  $CP^{n-1}$  Model”, *Nucl. Phys.* **B222** (1983) 45.
- [27] D. Morrison and R. Plesser, “Summing the Instantons: Quantum Cohomology and Mirror Symmetry in Toric Varieties”, DUKE-TH-94-78, IASSNS-HEP-94/82, hep-th/94122236.
- [28] P. Candelas, X. de la Ossa, P. Green, and L. Parkes, “A pair of Calabi-Yau Manifolds as an Exactly Soluble Conformal Field Theory”, *Nucl. Phys.* **B359** (1991) 21.
- [29] P. Candelas, “Yukawa Couplings of (2,1) Forms”, *Nucl. Phys.* **B298** (1988) 458.
- [30] C. Vafa, “Topological Landau-Ginzburg Models”, *Mod. Phys. Lett.* **A6** (1991) 337.
- [31] E. Silverstein, “Miracle at the Gepner Point”, PUPT-1519, hep-th/9503150.
- [32] P. Aspinwall and B. Greene, “On the Geometrical Interpretation of N=2 Superconformal Theories”, CLNS-94-1299, hep-th/9409110.